Poincaré and Weak Poincaré Inequalities for the Mixed Poisson Measures

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Abstract

By using the Mecke identity, we study a class of birth-death type Dirichlet forms associated with the mixed Poisson measure. Both Poincaré and weak Poincaré inequalities are established, while another Poincaré type inequality is disproved under some reasonable assumptions.

AMS subject Classification: 28C20, 60J80.

Keywords: Mixed Poisson measure, configuration space, birth-death process, Poincaré inequality, weak Poincaré inequality.

1 Introduction

Let X be a locally compact Polish space with Borel σ-field $\mathscr{B}(X)$ and let μ be a Radon measure on $(X, \mathscr{B}(X))$. Let Γ_X denote the space of all $\mathbb{Z}_+ \cup \{\infty\}$-valued Radon measures on X. The set of all $\gamma \in \Gamma_X$ such that $\gamma(\{x\}) \in \{0, 1\}$ for all $x \in X$ is called the configuration space over X. For simplicity, we also call Γ_X the configuration space over X. Endow Γ_X with the vague topology, i.e. the weakest topology on Γ_X such that the mapping

$$\Gamma_X \ni \gamma \mapsto \langle \gamma, f \rangle := \gamma(f) = \int_X f \, \mathrm{d}\gamma \in \mathbb{R}$$

is continuous for every $f \in C_0(X)$. Here $C_0(X)$ denotes the space of all continuous functions on X having compact support. Denote by $\mathscr{B}(\Gamma_X)$ the corresponding Borel σ-field on Γ_X. The (pure) Poisson measure with intensity μ, denoted by π_μ, is the unique

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probability measure on $(\Gamma_X, \mathscr{B}(\Gamma_X))$ with Laplace transform given by

$$\pi_\mu \left(\mathrm{e}^{\langle \cdot, f \rangle} \right) = \exp \left[\int_X (\mathrm{e}^f - 1) \, \mathrm{d}\mu \right], \quad f \in L^1(\mu) \cap L^\infty(\mu). \tag{1.1}$$

Another characteristic of the Poisson measure π_μ is that for any disjoint sets $A_1, \cdots, A_n \in \mathscr{B}(X)$ with $\mu(A_i) < \infty$, $1 \le i \le n$,

$$\pi_\mu\big(\{\gamma \in \Gamma_X;\ \gamma(A_i) = k_i, 1 \le i \le n\}\big) = \prod_{i=1}^n \mathrm{e}^{-\mu(A_i)} \frac{\mu(A_i)^{k_i}}{k_i!}, \quad k_i \in \mathbb{Z}_+,\ 1 \le i \le n.$$

We remark that for $\mu = 0$, π_μ is just the Dirac measure on $(\Gamma_X, \mathscr{B}(\Gamma_X))$ with total mass in the empty configuration $\gamma = 0$ (i.e. the zero measure on X). We refer to e.g. [1, 12] for a detailed discussion of the construction of the Poisson measure on configuration space.

Recall the birth-death type Dirichlet form associated with the Poisson measure:

$$\begin{aligned} \mathscr{E}_{bd}(F, G) &= \int_{\Gamma_X \times X} \big(F(\gamma + \delta_x) - F(\gamma)\big)\big(G(\gamma + \delta_x) - G(\gamma)\big) \, \pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}x), \\ \mathscr{D}(\mathscr{E}_{bd}) &= \{F \in L^2(\pi_\mu);\ \mathscr{E}_{bd}(F, F) < \infty\}. \end{aligned}$$

It is well known that $(\mathscr{E}_{bd}, \mathscr{D}(\mathscr{E}_{bd}))$ does not satisfy the log-Sobolev inequality (cf. [13]). In the paper [15], Wu established the Poincaré and L^1 log-Sobolev inequalities for $(\mathscr{E}_{bd}, \mathscr{D}(\mathscr{E}_{bd}))$ by exploring the martingale representation (the counterpart on Poisson space of the Clark-Ocône formula over Wiener space). Recently, [14] presented a new proof of the Poincaré inequality for $(\mathscr{E}_{bd}, \mathscr{D}(\mathscr{E}_{bd}))$. Following the line of [14], the authors of this paper confirmed the L^1 log-Sobolev inequality for $(\mathscr{E}_{bd}, \mathscr{D}(\mathscr{E}_{bd}))$ again in [2].

The central aim of the present paper is to extend the known results concerning Poincaré type inequalities for the birth-death type Dirichlet forms associated with the Poisson measure to the mixed Poisson measure case.

The mixed Poisson measure is defined by

$$\pi_{\lambda,\mu} = \int_{\mathbb{R}^+} \pi_{s\mu} \, \lambda(\mathrm{d}s),$$

where λ is a probability measure on $\mathbb{R}^+ := [0, \infty)$. See [1, 10] for more details on the mixed Poisson measure. In particular, if $\lambda = \delta_1$, then $\pi_{\lambda,\mu}$ reduces to the (pure) Poisson measure π_μ. Another interesting case is the fractional Poisson measure (cf. [8]).

Now consider the quadric form

$$\begin{aligned} \mathscr{E}(F, G) &:= \int_{\Gamma_X \times X \times \mathbb{R}^+} \big(F(\gamma + \delta_x) - F(\gamma)\big)\big(G(\gamma + \delta_x) - G(\gamma)\big) \, \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s), \\ \mathscr{D}(\mathscr{E}) &:= \{F \in L^2(\pi_{\lambda,\mu});\ \mathscr{E}(F, F) < \infty\}. \end{aligned}$$

According to Proposition 2.2 below, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a conservative symmetric Dirichlet form on $L^2(\pi_{\lambda,\mu})$ provided $\int_{\mathbb{R}^+} s \, \lambda(\mathrm{d}s) < \infty$. Now we are ready to present our main results.

Theorem 1.1. *If $\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s) < \infty$, $\mu(X) < \infty$ and there exists a constant $C \geq 0$ such that*

$$\lambda(f^2) - \lambda(f)^2 \leq C \int_{\mathbb{R}^+} s|f'(s)|^2\,\lambda(\mathrm{d}s), \quad f \in C_b^1(\mathbb{R}^+), \tag{1.2}$$

then

$$\pi_{\lambda,\mu}(F^2) - \pi_{\lambda,\mu}(F)^2 \leq \big(1 + C\mu(X)\big)\mathscr{E}(F,F), \quad F \in \mathscr{D}(\mathscr{E}).$$

Remark 1.2. *When $\lambda = \delta_1$, obviously (1.2) holds with $C = 0$, and so without the assumption that $\mu(X) < \infty$ Theorem 1.1 reduces to the Poincaré inequality for the Poisson measure (see [15, Remark 1.4]) with the sharp constant 1 (see [3]).*

Next, we consider the weak Poincaré inequality, which was introduced in [11] to describe the non-exponential convergence rates of Markov semigroups. Let

$$\|f\|_u = \sup_{s \in \mathbb{R}^+} |f(s)|, \quad \text{and} \quad \|F\|_u = \sup_{\gamma \in \Gamma_X} |F(\gamma)|,$$

where f and F are functions on $\mathbb{R}^+$ and Γ_X, respectively.

Theorem 1.3. *If $\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s) < \infty$, $\mu(X) < \infty$ and there exists $\alpha : (0,\infty) \to (0,\infty)$ such that*

$$\lambda(f^2) - \lambda(f)^2 \leq \alpha(r) \int_{\mathbb{R}^+} s|f'(s)|^2\,\lambda(\mathrm{d}s) + r\|f\|_u^2, \quad r > 0, f \in C_b^1(\mathbb{R}^+), \tag{1.3}$$

then

$$\pi_{\lambda,\mu}(F^2) - \pi_{\lambda,\mu}(F)^2 \leq \big(1 + \mu(X)\alpha(r)\big)\mathscr{E}(F,F) + r\|F\|_u^2, \quad r > 0, F \in \mathscr{D}(\mathscr{E}).$$

Finally, we point out that under some assumptions, the following Poincaré type inequality fails:

$$\pi_{\lambda,\mu}(F^2) \leq C_1\mathscr{E}(F,F) + C_2\pi_{\lambda,\mu}(|F|)^2, \quad F \in \mathscr{D}(\mathscr{E}), \tag{1.4}$$

where C_1 and C_2 are constants.

Proposition 1.4. *Assume that $\int_{\mathbb{R}^+} s^2\,\lambda(\mathrm{d}s) \in (0,\infty)$, and there exists a sequence $\{A_n\}_{n\geq 1} \subset \mathscr{B}(X)$ such that $\mu(A_n) > 0$ for every $n \geq 1$ but $\mu(A_n) \to 0$ as $n \to \infty$ (it is the case if μ does not have atom). If $C_1 < 1$, then (1.4) fails for any C_2.*

The remainder of the paper is organized as follows. In Section 2, we discuss the birth-death type Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. The proofs of the main results are addressed in Section 3.

2 Birth-death type Dirichlet form

In this section, we shall follow the line of [3] to characterize the form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. First of all, recall the Mecke identity [6] (see also [10]):

$$\int_{\Gamma_X \times X} H(\gamma + \delta_x, x)\, \pi_\mu(\mathrm{d}\gamma)\mu(\mathrm{d}x) = \int_{\Gamma_X \times X} H(\gamma, x)\, \gamma(\mathrm{d}x)\pi_\mu(\mathrm{d}\gamma) \tag{2.1}$$

holds for any measurable function H on $\Gamma_X \times X$ such that one of the above integrals exists.

For $A \in \mathscr{B}(\Gamma_X)$, let

$$\tilde{A} = \{(\gamma, x) \in \Gamma_X \times X;\ \gamma + \delta_x \in A\}.$$

We first prove the following quasi-invariant property of the mapping $\gamma \mapsto \gamma + \delta_x$.

Lemma 2.1. *If* $\pi_{\lambda,\mu}(A) = 0$, *then*

$$\int_{\tilde{A} \times \mathbb{R}^+} \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s) = 0.$$

Proof. By the Mecke identity (2.1) for $H(\gamma, \cdot) = 1_A(\gamma)$ and using the definition of $\pi_{\lambda,\mu}$, we have

$$\begin{aligned}
\int_{\tilde{A} \times \mathbb{R}^+} \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s) &= \int_{\mathbb{R}^+} \lambda(\mathrm{d}s) \int_{\Gamma_X \times X} 1_A(\gamma + \delta_x)\, \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x) \\
&= \int_{\mathbb{R}^+} \lambda(\mathrm{d}s) \int_{\Gamma_X \times X} 1_A(\gamma)\, \gamma(\mathrm{d}x)\pi_{s\mu}(\mathrm{d}\gamma) \\
&= \int_{\mathbb{R}^+} \lambda(\mathrm{d}s) \int_A \gamma(X)\, \pi_{s\mu}(\mathrm{d}\gamma) \\
&= \int_A \gamma(X)\, \pi_{\lambda,\mu}(\mathrm{d}\gamma) = 0.
\end{aligned}$$

□

Define the family of cylindrical functions

$$\mathscr{F}_\mu^C = \big\{\gamma \mapsto f(\gamma(h_1), \cdots, \gamma(h_m));\ m \geq 1, f \in C_b^1(\mathbb{R}^m), h_i \in L^1(\mu) \cap L^\infty(\mu)\big\}.$$

Proposition 2.2. *Assume that* $\int_{\mathbb{R}^+} s\, \lambda(\mathrm{d}s) < \infty$. *Then* $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ *is a conservative symmetric Dirichlet form on* $L^2(\pi_{\lambda,\mu})$ *with* $\mathscr{D}(\mathscr{E}) \supset \mathscr{F}_\mu^C$.

Proof. Due to Lemma 2.1, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is well-defined on $L^2(\pi_{\lambda,\mu})$; that is, the value of $\mathscr{E}(F,G)$ does not depend on $\pi_{\lambda,\mu}$-versions of F and G. Note that $\mathscr{F}_\mu^C$ is dense in $L^2(\pi_{\lambda,\mu})$ and by the definition of $\mathscr{E}$, the normal contractivity property is trivial. Therefore, we only need to prove $\mathscr{F}_\mu^C \subset \mathscr{D}(\mathscr{E})$ and the closed property of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. Let's prove these two points separately.

(1) For any $h \in L^1(\mu) \cap L^\infty(\mu)$, noting that

$$\begin{aligned}\int_{\Gamma_X} |\gamma(h)|\, \pi_{\lambda,\mu}(\mathrm{d}\gamma) &\le \int_{\mathbb{R}^+} \lambda(\mathrm{d}s) \int_{\Gamma_X} \gamma(|h|)\, \pi_{s\mu}(\mathrm{d}\gamma)\\ &= \mu(|h|) \int_{\mathbb{R}^+} s\, \lambda(\mathrm{d}s) < \infty,\end{aligned}$$

one has $\gamma(h) \in \mathbb{R}$ for $\pi_{\lambda,\mu}$-a.e. $\gamma \in \Gamma_X$. Therefore,

$$F(\gamma) := f(\gamma(h_1), \cdots, \gamma(h_m)) \in \mathscr{F}_\mu^C$$

is well-defined in $L^2(\pi_{\lambda,\mu})$. By the Mean Value Theorem and noting that $h_i \in L^1(\mu) \cap L^\infty(\mu)$, we arrive at

$$\begin{aligned}\mathscr{E}(F,F) &= \int_{\Gamma_X \times X \times \mathbb{R}^+} \big[f(\gamma(h_1) + h_1(x), \cdots, \gamma(h_m) + h_m(x))\\ &\qquad - f(\gamma(h_1), \cdots, \gamma(h_m))\big]\, \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s)\\ &\le \|\nabla f\|_\infty^2 \int_{\Gamma_X \times X \times \mathbb{R}^+} \sum_{i=1}^m h_i(x)^2\, \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s)\\ &= \|\nabla f\|_\infty^2 \left(\sum_{i=1}^m \mu(h_i^2)\right) \int_{\mathbb{R}^+} s\, \lambda(\mathrm{d}s)\\ &\le \|\nabla f\|_\infty^2 \left(\sum_{i=1}^m \|h_i\|_\infty \mu(|h_i|)\right) \int_{\mathbb{R}^+} s\, \lambda(\mathrm{d}s)\\ &< \infty.\end{aligned}$$

Thus, $\mathscr{F}_\mu^C \subset \mathscr{D}(\mathscr{E})$.

(2) Let $\{F_n\}_{n\ge1}$ be an $\mathscr{E}_1^{1/2}$-Cauchy sequence, where

$$\mathscr{E}_1^{1/2}(F,F) := \sqrt{\mathscr{E}(F,F) + \pi_{\lambda,\mu}(|F|^2)}, \quad F \in \mathscr{D}(\mathscr{E}).$$

Since $\{F_n\}_{n\ge1}$ is a Cauchy sequence in $L^2(\pi_{\lambda,\mu})$, there exists $F \in L^2(\pi_{\lambda,\mu})$ such that $F_n \to F$ in $L^2(\pi_{\lambda,\mu})$. Consequently, we can choose a subsequence $\{F_{n_k}\}_{k\ge1}$ such that

$F_{n_k} \to F$ $\pi_{\lambda,\mu}$-a.e. Then it follows from Lemma 2.1 that $F_{n_k}(\gamma+\delta_x) \to F(\gamma+\delta_x)$ for $\left(\int_{\mathbb{R}^+} \pi_{s\mu} s\lambda(\mathrm{d}s)\right) \times \mu$-a.e. $(\gamma, x) \in \Gamma_X \times X$. So, we obtain from the Fatou Lemma that

$$
\begin{aligned}
&\mathscr{E}(F_n - F, F_n - F)\\
&= \int_{\Gamma_X \times X \times \mathbb{R}^+} \liminf_{k\to\infty} \left[(F_n - F_{n_k})(\gamma+\delta_x) - (F_n - F_{n_k})(\gamma)\right]^2 \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s)\\
&\le \liminf_{k\to\infty} \mathscr{E}(F_n - F_{n_k}, F_n - F_{n_k}).
\end{aligned}
$$

Since $\{F_n\}_{n\ge 1}$ is an $\mathscr{E}_1^{1/2}$-Cauchy sequence, this implies that

$$
\lim_{n\to\infty} \mathscr{E}_1^{1/2}(F_n - F, F_n - F) = 0. \tag{2.2}
$$

On the other hand, using the Fatou Lemma again, we get

$$
\begin{aligned}
\mathscr{E}(F, F) &= \int_{\Gamma_X \times X \times \mathbb{R}^+} \liminf_{k\to\infty} (F_{n_k}(\gamma+\delta_x) - F_{n_k}(\gamma))^2 \pi_{s\mu}(\mathrm{d}\gamma) s\mu(\mathrm{d}x)\lambda(\mathrm{d}s)\\
&\le \liminf_{k\to\infty} \mathscr{E}(F_{n_k}, F_{n_k})\\
&< \infty
\end{aligned}
$$

since $\{F_{n_k}\}_{k\ge 1}$ is an $\mathscr{E}^{1/2}$-Cauchy sequence. Combining this with (2.2), we conclude that $F \in \mathscr{D}(\mathscr{E})$ and $F_n \to F$ in $\mathscr{D}(\mathscr{E})$ as $n \to \infty$. Hence, $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is closed and the proof is now completed. □

Now we move on to study the regularity of the Dirichlet form. Note that Γ_X is a Polish space under the vague topology (see [9, Proposition 3.17]). Since the probability measure $\pi_{\lambda,\mu}$ on the Polish space Γ_X is tight, we can choose an increasing sequence $\{K_n\}_{n\ge 1}$ consisting of compact subsets of Γ_X such that $\pi_{\lambda,\mu}(K_n^c) \le 1/n$ for any $n \ge 1$. Then $\pi_{\lambda,\mu}$ has full measure on $\Gamma_X^\mu := \bigcup_{n=1}^\infty K_n$, which is a locally compact separable metric space.

Proposition 2.3. *If $\mu(X) < \infty$ and $\int_{\mathbb{R}^+} s\lambda(\mathrm{d}s) < \infty$, then $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a regular Dirichlet form on $L^2(\Gamma_X^\mu; \pi_{\lambda,\mu})$.*

Proof. Since $\mu(X) < \infty$, it is easy to see that $\mathscr{B}_b(\Gamma_X^\mu) \subset \mathscr{D}(\mathscr{E})$, where $\mathscr{B}_b(\Gamma_X^\mu)$ is the set of all bounded measurable functions on Γ_X^μ. In particular, $C_0(\Gamma_X^\mu) \subset \mathscr{D}(\mathscr{E})$. Thus, it suffices to prove that $C_0(\Gamma_X^\mu)$ is dense in $\mathscr{D}(\mathscr{E})$ w.r.t. the $\mathscr{E}_1$-norm, i.e. for any $F \in \mathscr{D}(\mathscr{E})$, one may find a sequence $\{F_n\}_{n\ge 1} \subset C_0(\Gamma_X^\mu)$ such that $\mathscr{E}_1(F_n - F, F_n - F) \to 0$ as $n \to \infty$.

Since $\mathscr{B}_b(\Gamma_X^\mu) \cap \mathscr{D}(\mathscr{E})$ is dense in $\mathscr{D}(\mathscr{E})$ (see e.g. [5, Proposition I.4.17]), we may assume that $F \in \mathscr{B}_b(\Gamma_X^\mu)$. Moreover, since $C_0(\Gamma_X^\mu)$ is dense in $L^2(\Gamma_X^\mu; \pi_{\lambda,\mu})$, we may find a sequence $\{F_n\}_{n\ge 1} \subset C_0(\Gamma_X^\mu)$ such that $\sup_{n\in\mathbb{N}} \|F_n\|_{L^\infty(\pi_{\lambda,\mu})} \le \|F\|_{L^\infty(\pi_{\lambda,\mu})}$ and $\pi_{\lambda,\mu}(|F_n - F|^2) \to 0$ as $n \to \infty$. Without loss of generality, we assume furthermore that $F_n \to F$ $\pi_{\lambda,\mu}$-a.e. By

Lemma 2.1, $F_n(\gamma+\delta_x) \to F(\gamma+\delta_x)$ and $(F_n-F)^2(\gamma+\delta_x) \le (\|F_n\|_{L^\infty(\pi_{\lambda,\mu})}+\|F\|_{L^\infty(\pi_{\lambda,\mu})})^2 \le 4\|F\|^2_{L^\infty(\pi_{\lambda,\mu})}$ for $\left(\int_{\mathbb{R}^+} \pi_{s\mu} s\lambda(\mathrm{d}s)\right) \times \mu$-a.e. $(\gamma, x) \in \Gamma_X \times X$.

Note that (we do not have to distinguish integrals on Γ_X^μ and Γ_X since $\pi_{\lambda,\mu}(\Gamma\backslash\Gamma_X^\mu) = 0$)

$$\begin{aligned}&\mathscr{E}(F_n - F, F_n - F)\\ &\le 2\int_{\Gamma_X\times X\times\mathbb{R}^+} (F_n - F)^2(\gamma + \delta_x)\, \pi_{s\mu}(\mathrm{d}\gamma)s\mu(\mathrm{d}x)\lambda(\mathrm{d}s)\\ &\quad + 2\int_{\Gamma_X\times X\times\mathbb{R}^+} (F_n - F)^2(\gamma)\, \pi_{s\mu}(\mathrm{d}\gamma)s\mu(\mathrm{d}x)\lambda(\mathrm{d}s).\end{aligned}$$

By the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \mathscr{E}(F_n - F, F_n - F) = 0.$$

Combining this with $\pi_{\lambda,\mu}(|F_n - F|^2) \to 0$, we conclude that

$$\lim_{n\to\infty} \mathscr{E}_1(F_n - F, F_n - F) = 0,$$

which completes the proof. □

3 Proofs of Theorems 1.1 and 1.3 and Proposition 1.4

To prove Theorem 1.1, let's first make some preparations.

Lemma 3.1. *If $\mu(X) < \infty$, then for any $s, t \in \mathbb{R}^+$, $\pi_{s\mu}$ and $\pi_{t\mu}$ are mutually absolutely continuous. More precisely,*

$$\frac{\mathrm{d}\pi_{t\mu}}{\mathrm{d}\pi_{s\mu}}(\gamma) = \exp\left[\gamma(X)\log\frac{t}{s} + (s-t)\mu(X)\right]$$

for $\pi_{t\mu}$-a.e. (and so also $\pi_{s\mu}$-a.e.) $\gamma \in \Gamma_X$.

Proof. For every $f \in L^1(\mu) \cap L^\infty(\mu)$, it follows from (1.1) that

$$\begin{aligned}\int_{\Gamma_X} \mathrm{e}^{\gamma(f)}\, \pi_{t\mu}(\mathrm{d}\gamma) &= \exp\left[t\mu(\mathrm{e}^f - 1)\right]\\ &= \mathrm{e}^{(s-t)\mu(X)} \exp\left[s\mu(\mathrm{e}^{f+\log\frac{t}{s}} - 1)\right]\\ &= \mathrm{e}^{(s-t)\mu(X)} \int_{\Gamma_X} \mathrm{e}^{\gamma(f+\log\frac{t}{s})}\, \pi_{s\mu}(\mathrm{d}\gamma)\\ &= \int_{\Gamma_X} \mathrm{e}^{\gamma(f)} \exp\left[\gamma(X)\log\frac{t}{s} + (s-t)\mu(X)\right]\, \pi_{s\mu}(\mathrm{d}\gamma),\end{aligned}$$

from which we obtain the desired result. □

Let $\mathscr{B}_b(\Gamma_X)$ be the family of bounded measurable functions on Γ_X. The following result, proved in [4] (see also [7] for more general result), is crucial for the proof of Theorems 1.1 and 1.3. Here we shall give another easy proof, which can be regarded as an interesting application of Lemma 3.1.

Lemma 3.2. *If $\mu(X) < \infty$, then for every $F \in \mathscr{B}_b(\Gamma_X)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\pi_{s\mu}(F) = \int_{\Gamma_X \times X} \big(F(\gamma+\delta_x) - F(\gamma)\big)\, \pi_{s\mu}(\mathrm{d}\gamma)\mu(\mathrm{d}x), \quad s \in \mathbb{R}^+.$$

Proof. By Lemma 3.1, for any $s \in \mathbb{R}^+$ and $\varepsilon \geq -s$, one has

$$\frac{\mathrm{d}\pi_{(s+\varepsilon)\mu}}{\mathrm{d}\pi_{s\mu}}(\gamma) = \exp\left[\gamma(X)\log\frac{s+\varepsilon}{s} - \varepsilon\mu(X)\right]$$

for $\pi_{s\mu}$-a.e. $\gamma \in \Gamma_X$. Combining this with the dominated convergence theorem, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\pi_{s\mu}(F) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\int_{\Gamma_X} F(\gamma)\,\pi_{(s+\varepsilon)\mu}(\mathrm{d}\gamma) - \int_{\Gamma_X} F(\gamma)\,\pi_{s\mu}(\mathrm{d}\gamma)\right\}\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma_X} F(\gamma)\left\{\exp\left[\gamma(X)\log\frac{s+\varepsilon}{s} - \varepsilon\mu(X)\right] - 1\right\}\pi_{s\mu}(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X} F(\gamma)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{\exp\left[\gamma(X)\log\frac{s+\varepsilon}{s} - \varepsilon\mu(X)\right] - 1\right\}\pi_{s\mu}(\mathrm{d}\gamma)\\
&= \int_{\Gamma_X} F(\gamma)\left(\frac{1}{s}\gamma(X) - \mu(X)\right)\pi_{s\mu}(\mathrm{d}\gamma)\\
&= \frac{1}{s}\int_{\Gamma_X\times X} F(\gamma)\,\gamma(\mathrm{d}x)\pi_{s\mu}(\mathrm{d}\gamma) - \int_{\Gamma_X\times X} F(\gamma)\,\pi_{s\mu}(\mathrm{d}\gamma)\mu(\mathrm{d}x)\\
&= \int_{\Gamma_X\times X}\big(F(\gamma+\delta_x) - F(\gamma)\big)\,\pi_{s\mu}(\mathrm{d}\gamma)\mu(\mathrm{d}x),
\end{aligned}$$

where in the last step we have used the Mecke identity (2.1) for $H(\gamma,\cdot) = F(\gamma)$. □

Proof of Theorem 1.1. By an approximation argument, we may and do assume that $F \in \mathscr{B}_b(\Gamma_X)$. Recalling the Poincaré inequality for birth-death type Dirichlet form w.r.t. the Poisson measure (see [15, Remark 1.4]), we have

$$\pi_{s\mu}(F^2) \leq \pi_{s\mu}(F)^2 + \int_{\Gamma_X\times X}(F(\gamma+\delta_x) - F(\gamma))^2\,\pi_{s\mu}(\mathrm{d}\gamma)s\mu(\mathrm{d}x), \quad s \in \mathbb{R}^+.$$

This yields that

$$\begin{aligned}
\pi_{\lambda,\mu}(F^2) - \pi_{\lambda,\mu}(F)^2 &= \int_{\mathbb{R}^+}\pi_{s\mu}(F^2)\,\lambda(\mathrm{d}s) - \left(\int_{\mathbb{R}^+}\pi_{s\mu}(F)\,\lambda(\mathrm{d}s)\right)^2\\
&\leq \mathscr{E}(F,F) + \int_{\mathbb{R}^+}\pi_{s\mu}(F)^2\,\lambda(\mathrm{d}s) - \left(\int_{\mathbb{R}^+}\pi_{s\mu}(F)\,\lambda(\mathrm{d}s)\right)^2.
\end{aligned} \tag{3.1}$$

Applying (1.2) with $f(s) = \pi_{s\mu}(F)$, and using Lemma 3.2 and the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^+} \pi_{s\mu}(F)^2 \,\lambda(\mathrm{d}s) - \left(\int_{\mathbb{R}^+} \pi_{s\mu}(F)\,\lambda(\mathrm{d}s)\right)^2 \\
&\le C \int_{\mathbb{R}^+} s \left|\frac{\mathrm{d}}{\mathrm{d}s}\pi_{s\mu}(F)\right|^2 \lambda(\mathrm{d}s) \\
&= C \int_{\mathbb{R}^+} s \left(\int_{\Gamma_X \times X} (F(\gamma+\delta_x) - F(\gamma))\,\pi_{s\mu}(\mathrm{d}\gamma)\mu(\mathrm{d}x)\right)^2 \lambda(\mathrm{d}s) \\
&\le C\mu(X)\mathscr{E}(F,F).
\end{aligned}$$

Combining this with (3.1), the desired Poincaré inequality for $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ follows. □

Proof of Theorem 1.3. Similarly as in the proof of Theorem 1.1, we assume that $F \in \mathscr{B}_b(\Gamma_X)$. Since $\|\pi_{\cdot\mu}(F)\|_u \le \|F\|_u$, it holds from (1.3), Lemma 3.2 and the Cauchy-Schwartz inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^+} \pi_{s\mu}(F)^2 \,\lambda(\mathrm{d}s) - \left(\int_{\mathbb{R}^+} \pi_{s\mu}(F)\,\lambda(\mathrm{d}s)\right)^2 \\
&\le \alpha(r) \int_{\mathbb{R}^+} s \left|\frac{\mathrm{d}}{\mathrm{d}s}\pi_{s\mu}(F)\right|^2 \lambda(\mathrm{d}s) + r\|F\|_u^2 \\
&\le \mu(X)\alpha(r)\mathscr{E}(F,F) + r\|F\|_u^2, \quad r > 0.
\end{aligned}$$

Hence we finish the proof by combining this with (3.1). □

To prove Proposition 1.4, we need the following fundamental lemma. We include a simple proof for completeness.

Lemma 3.3. *Assume that $\int_{\mathbb{R}^+} s^2 \lambda(\mathrm{d}s) < \infty$. Then for $F(\gamma) = \gamma(f)$, where $f \in L^1(\mu) \cap L^\infty(\mu)$ and $f \ge 0$ μ-a.e., we have*

$$\pi_{\lambda,\mu}(F) = \mu(f) \int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s), \quad \pi_{\lambda,\mu}(F^2) = \mu(f)^2 \int_{\mathbb{R}^+} s^2\,\lambda(\mathrm{d}s) + \mu(f^2)\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s).$$

Proof. Let $\varepsilon \le 0$. It follows from (1.1) that

$$\int_{\Gamma_X} \mathrm{e}^{\varepsilon\gamma(f)}\,\pi_{\lambda,\mu}(\mathrm{d}\gamma) = \int_{\mathbb{R}^+} \exp\left[s\int_X \left(\mathrm{e}^{\varepsilon f} - 1\right)\,\mathrm{d}\mu\right] \lambda(\mathrm{d}s).$$

Since $\varepsilon \le 0$, $f \ge 0$ and $\int_{\mathbb{R}^+} s^2 \lambda(\mathrm{d}s) < \infty$, this implies that

$$\int_{\Gamma_X} \mathrm{e}^{\varepsilon\gamma(f)}\gamma(f)\,\pi_{\lambda,\mu}(\mathrm{d}\gamma) = \int_{\mathbb{R}^+} s\exp\left[s\int_X \left(\mathrm{e}^{\varepsilon f} - 1\right)\,\mathrm{d}\mu\right] \lambda(\mathrm{d}s) \times \int_X f\mathrm{e}^{\varepsilon f}\,\mathrm{d}\mu$$

and

$$\int_{\Gamma_X} \mathrm{e}^{\varepsilon\gamma(f)}\gamma(f)^2\,\pi_{\lambda,\mu}(\mathrm{d}\gamma) = \int_{\mathbb{R}^+} s^2 \exp\left[s\int_X \left(\mathrm{e}^{\varepsilon f}-1\right)\,\mathrm{d}\mu\right]\lambda(\mathrm{d}s)\times\left(\int_X f\mathrm{e}^{\varepsilon f}\,\mathrm{d}\mu\right)^2$$
$$+\int_{\mathbb{R}^+} s\exp\left[s\int_X \left(\mathrm{e}^{\varepsilon f}-1\right)\,\mathrm{d}\mu\right]\lambda(\mathrm{d}s)\times\int_X f^2\mathrm{e}^{\varepsilon f}\,\mathrm{d}\mu.$$

Now we finish the proof by taking $\varepsilon = 0$ in these two equalities. □

Proof of Proposition 1.4. Suppose that (1.4) holds with $C_1 < 1$ and some C_2. Fix a nonnegative $f \in L^1(\mu)\cap L^\infty(\mu)$ and let $F(\gamma) = \gamma(f)$. Then we deduce from (1.4) and Lemma 3.3 that

$$\mu(f)^2\int_{\mathbb{R}^+} s^2\,\lambda(\mathrm{d}s) + \mu(f^2)\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s) \le C_1\mu(f^2)\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s) + C_2\mu(f)^2\left(\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s)\right)^2,$$

which yields that

$$\left[(1-C_1)\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s)\right]\mu(f^2) \le \left[C_2\left(\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s)\right)^2 - \int_{\mathbb{R}^+} s^2\,\lambda(\mathrm{d}s)\right]\mu(f)^2$$

holds for any nonnegative $f \in L^1(\mu)\cap L^\infty(\mu)$. Noting that $(1-C_1)\int_{\mathbb{R}^+} s\,\lambda(\mathrm{d}s) \in (0,\infty)$, this is impossible according to the assumption that we can choose a sequence $\{A_n\}_{n\ge1}$ such that $\mu(A_n) > 0$ but $\mu(A_n) \to 0$ as $n \to \infty$. □

References

[1] Aibeverio S, Kondratiev Y G, Röckner M. Analysis and geometry on configuration spaces. J Funct Anal, 1998, 154: 444-500.

[2] Deng C S, Song Y H. An elementary proof of the L^1 log-Sobolev inequality for Poisson point processes. Statist Probab Lett, 2011, 81: 1458-1462.

[3] Deng C S, Wang F Y. Exponential convergence rates of second quantization semigroups and applications. Accessible on arXiv:1012.5689.

[4] Houdré C, Privault N. Isoperimetric and related bounds on configuration spaces. Statist Probab Lett, 2008, 78: 2154-2164.

[5] Ma Z M, Röckner M. Introduction to the Theory of (Non-Symmetric) Dirichlet Forms. Berlin: Springer-Verlag, 1992.

[6] Mecke J. Stationaire Zufällige Maβe auf lokalkompakten abelschen Gruppen. Z Wahrsch verw Geb, 1967, 9: 36-58.

[7] Møller J, Zuyev S. Gamma-type results and other related properties of Poisson processes. Adv Appl Prob, 1996, 28: 662-673.

[8] Oliveira M J, Ouerdiane H, Silva J L, Vilela Mendes R. The fractional Poisson measure in infinite dimensions. Accessible on arXiv:1002.2124v1.

[9] Resnick S I. Extreme Values, Regular Variation, and Point Processes. Berlin, Heidelberg, New York: Springer, 1987.

[10] Röckner M. Stochastic analysis on configuration spaces: basic ideas and recent results. In: New Directions in Dirichlet forms. AMS/IP Stud Adv Math 8, 1998, 157-231.

[11] Röckner M, Wang F Y. Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. J Funct Anal, 2001, 185: 564-603.

[12] Shimomura H. Poisson measures on configuration space and unitary representation of the group of diffeomorphisms. J Math Kyoto Univ, 1994, 34: 599-614.

[13] Surgailis D. On the multiple Poisson stochastic integrals and associated Markov semigroups. Probab Math Statist, 1984, 3: 217-239.

[14] Wang F Y, Yuan C. Poincaré inequality on the path space of Poisson point processes. J Theor Probab, 2010, 23: 824-833.

[15] Wu L. A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. Probab Theory Relat Fields, 2000, 118: 427-438.